Quiz

① What is your name?

② Define "$\lim a_n = L$".

③ State the Axiom of Completeness.

Recall what we know about series:

① If $\sum a_k$ is a series, it's really the sequence of partial sums (s_n), where $s_n = \sum_{k=1}^{n} a_k$.

② Geometric series $\sum a \cdot r^k$

$$s_n = \sum_{k=0}^{n} a r^k = a\,\frac{1-r^{n+1}}{1-r}$$

Converges $\Leftrightarrow |r|<1$.

$$|r|<1 \Rightarrow \lim s_n = \sum_{k=0}^{\infty} a\cdot r^k = \frac{a}{1-r}.$$

(3) Harmonic Series $\sum \frac{1}{k}$ diverges.

(4) CC for series:

$\sum a_k$ converges $\iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. if $k \geq n \geq N$, then $\left|\sum_{j=n}^{k} a_j\right| < \varepsilon$.

New stuff.

(n^{th} term Lemma)

Lemma. If $\sum a_k$ converges, then $\lim a_n = 0$.

Remark: Converse is false — eg harmonic series has $a_n = \frac{1}{n}$, $\lim \frac{1}{n} = 0$, but it diverges.

Pf: Suppose $\sum a_k$ converges. By CC, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$ (let $k = n$)

$\Rightarrow \left|\sum_{j=n}^{n} a_j\right| < \varepsilon \iff |a_n| < \varepsilon$

$\Leftrightarrow |a_n - 0| < \varepsilon$. Thus $\lim a_n = 0$. $\blacksquare$

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eg $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{2}{n^2}\right)$ diverges,

since $\lim\left(1-\frac{2}{n^2}\right) \overset{ALT}{=}$

$1 - 2\cdot\left(\lim \frac{1}{n}\right)^2 = 1 - 2\cdot 0$

$= 1 \neq 0$.

Thus by n^{th} term Lemma,

it diverges.

Lemma (Eventually convergent series' converge)

The series $\sum a_k$ converges

$\Longleftrightarrow \exists N_{\in\mathbb{N}}$ s.t. $\displaystyle\sum_{k=N}^{\infty} a_k$ converges.

Restated: The sequence (s_n) converges

$\Longleftrightarrow \exists N \in \mathbb{N}$ s.t. (s_{n+N}) converges.

Pf. ($\Rightarrow$) Let $N \in \mathbb{N}$. Suppose $\sum a_k$ converges.

Then by CC, $\exists M \in \mathbb{N}$ s.t. $\forall$
$k \geq n \geq M$, $\left|\sum_{j=n}^{k} a_j\right| < \varepsilon$.
Then, it's also true if $k \geq n \geq M+N$.
Thus $\sum_{k=N}^{\infty} a_k$ converges. ✓

($\Leftarrow$) Suppose $N \in \mathbb{N}$, and suppose
$\sum_{k=N}^{\infty} a_k$ converges.

$$\sum_{k=N}^{\infty} a_k = \sum_{k=1}^{\infty} a_{k+N-1}$$

Then$^{(CC)}$ $\exists Q \in \mathbb{N}$ s.t. if $k \geq n \geq Q$
then $\left|\sum_{j=n}^{k} a_{j+N-1}\right| < \varepsilon$.

Then Let $P = Q + N$.
then if $k_2 \geq n_2 \geq P = Q + N$,
then $\left|\sum_{j=n_2}^{k_2} a_j\right| = \left|\sum_{b=n_2-N+1}^{k_2-N+1} a_{b+N-1}\right|$

since $k_2 - N + 1 \geq n_2 - N + 1 \geq (Q+N) - N + 1$
$= Q + 1 > Q$,
this is $< \varepsilon$.

Thus, $\sum a_j$ converges. $\boxed{}$

Using this lemma, we can use any of our convergence tests if the hypothesis holds "eventually". That is, the hypothesis holds starting with the N^{th} term for some fixed $N \in \mathbb{N}$.

Comparison test.

Suppose $0 \leq a_k \leq b_k \quad \forall k \in \mathbb{N}$ and suppose $\sum b_k$ converges. Then $\sum a_k$ converges also.

(Eventual version: Suppose $0 \leq a_k \leq b_k$ $\forall k \geq N$ for a fixed $N \in \mathbb{N}$.......)

Proof: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$\forall\ k \geq n \geq N$, $\left|\sum_{j=n}^{k} b_j\right| < \varepsilon$,

by CC. Then

$$0 \leq \left|\sum_{j=n}^{k} a_j\right| = \sum_{j=n}^{k} a_j \leq \sum_{j=n}^{k} b_j$$

$$= \left|\sum_{j=n}^{k} b_j\right| < \varepsilon.$$

By CC, $\sum a_k$ converges. $\boxtimes$

Thm. (Cauchy Condensation Test)

Suppose (a_n) is nonnegative and decreasing. Then $\sum a_k$ converges.

$$\Longleftrightarrow \quad \sum 2^k a_{2^k} \text{ converges.}$$

(pf. (In book – similar to proof that harmonic series diverges)).

Cor.: The series $\sum \frac{1}{n^p}$ (for fixed $p \in [0,\infty)$) converges $\Longleftrightarrow p > 1$. (P-series test).

(This means $\sum \frac{1}{n^{1.000002}}$ converges,

~~though~~ though $\sum \frac{1}{n}$ diverges.)

Pf: $\sum \frac{1}{k^p}$ converges $\left(\frac{1}{k^p}\right)_{k\geq 1}$ decreasing, nonneg.

$\iff \sum 2^k \frac{1}{(2^k)^p}$ converges

$\iff \sum_{k=1} \frac{2^k}{2^{kp}} = \sum \frac{1}{2^{kp-k}}$

$= \sum \frac{1}{(2^{(p-1)})^k} = \sum \left(\frac{1}{2^{(p-1)}}\right)^k$ conv.

This is a geometric series with

$a = \frac{1}{2^{p-1}}$, $r = \frac{1}{2^{p-1}}$

This converges

$\iff \frac{1}{2^{p-1}} < 1$

$\iff 2^{p-1} > 1$

$\iff p > 1$. ☒

Example Prove that

$$\sum_{n\geq 1} \frac{2n + \sin(4n^2)}{85n^{2.5}} \text{ converges.}$$

$$\leq \frac{2n-1}{85n^{2.5}} \leq \frac{2n+\sin(4n^2)}{85n^{2.5}} \leq \frac{2n+1}{85n^{2.5}} \leq$$

$$\Rightarrow \frac{2n-n}{85n^{2.5}} \leq \frac{2n+\sin(4n^2)}{85n^{2.5}} \leq \frac{2n+n}{85n^{2.5}}$$

$$\Rightarrow 0 \leq \frac{1}{85}\frac{1}{n^{1.5}} \leq \text{blah} \leq \frac{3}{85}\frac{1}{n^{1.5}}$$

Then, since $\sum \frac{3}{85}\frac{1}{n^{1.5}} = \frac{3}{85}\sum \frac{1}{n^{1.5}}$ converges, by the comparison test,

$$\sum \frac{2n+\sin(4n^2)}{85n^{2.5}} \text{ also converges.}$$

∎

Thm (Algebraic Series Thm)

If $\sum a_k$, $\sum b_k$ converge,

then ① $\sum(a_k+b_k)$ converges

& $\sum a_k+b_k = \sum a_k + \sum b_k$.

② $\sum(a_k-b_k)$ converges

& $\sum(a_k-b_k)=\sum a_k - \sum b_k$.

③ $\forall c \in \mathbb{R}$, $\sum c a_k$ converges

& $\sum c a_k = c \sum a_k$.

<u>Thm</u> (Absolute Convergence Thm).

If the series $\sum |a_k|$ converges, then $\sum a_k$ converges, and any rearrangement of the series converges to the same #.

(Eg: Rearrangement of $\sum \frac{1}{n^2}$

is $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{1^2}+\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{9^2}+\cdots$)

<u>Proof of first part</u>: Suppose $\sum|a_k|$ converges.

By CC, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $k \geq n \geq N$, then

$$\left| \sum_{j=n}^{k} |a_j| \right| < \varepsilon.$$

Then $\left| \sum_{j=1}^{k} a_j \right| \leq \sum_{j=1}^{k} |a_j| = \left| \sum_{j=1}^{k} |a_j| \right| < \varepsilon.$

Thus, by CC, $\sum a_k$ converges. $\boxtimes$

Alternating Series Test — Let (a_k) be a nonnegative, decreasing sequence such that $\lim a_k = 0$.

Then $\sum_{k=1} (-1)^k a_k$ converges. (and $\sum (-1)^{k+1} a_k$ converges)

called an "alternating series".

Example: $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

has $a_k = \frac{1}{k}$. nonneg ✓, decreasing ✓

$\lim \frac{1}{k} = 0$. ✓

$\Rightarrow$ it converges (Alternating Harmonic Series).

Riemann arrangement Thm.

If $\sum a_k$ is a series that converges but such that $\sum |a_k|$ diverges. [i.e. converges conditionally], then $\forall S \in \mathbb{R} \cup \{\pm\infty\}$. $\exists$ a rearrangement of the series that converges to S.
